

# Deformed parafermionic algebra from single-band tight-binding dynamics

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## Abstract

The quantum dynamics of a driven single-band tight-binding model with different boundary conditions is considered. The relation between the Hamiltonian describing the single-band tight-binding dynamics and the Hamiltonian of a discrete-charge mesoscopic quantum circuit is elucidated. It is shown that the former Hamiltonian, with Dirichlet boundary conditions, can be considered as a realization of the deformed parafermionic polynomial algebras.

**Keywords:** Deformed algebra, parafermionic, Tight-binding model, Discrete charge, mesoscopic circuit.

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## 1 Introduction

The dynamics of a quantum particle in a periodic potential under the effect of an external field is one of the most fascinating phenomena of quantum physics [1,2]. Under this condition, the electronic wave function displays so called Bloch Oscillations (BO) [3,4], the amplitude of which is proportional to the band width. A suitable orthogonal basis for investigating structures with periodic potential is using the localized states. These states are also called the Wannier-Stark states [5-9] and the nature of this states has a significant influence on the electronic transport properties of solids. The electronic BO was observed for the first time in semiconductor superlattices [10]. In [10,11], the first experimental observation of BO was reported. Recently, an increasing interest in the dynamics of BO can be observed [7,11-20].

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An explicit time dependence of physical properties can appear under several conditions. The most obvious case is when the external field depends on time. In principle, one has to define an initial state  $|\Psi(t=0)\rangle$  and solve the time-dependent Schrödinger equation. This is a partial differential equation with at least two variables which is separable in some simple cases [21,22], but rarely solved analytically.

In particular for the tight-binding model, some analytical expressions have been derived both for time-dependent and -independent fields [23-25]. Recently in the context of the tight-binding model, a treatment based on the dynamical Lie algebra is proposed in [26-29]. The time-dependent Hamiltonian which we will mainly consider here is the Hamiltonian of a finite system with Dirichlet boundary conditions defined by

$$\hat{H} = G \sum_{j=1}^{N-1} (|j\rangle\langle j+1| + |j+1\rangle\langle j|) + F(t) \sum_{j=1}^N j|j\rangle\langle j|, \quad (1)$$

where the ket  $|j\rangle$  represents a Wannier state located on the site  $j$ . The Hamiltonian (1) with infinite boundary conditions has been investigated for example in [15,27]. The Hamiltonian (1) can be treated as the Hamiltonian of a finite quantum wire, in tight-binding approximation, under the influence of a time-dependent electric field. This Hamiltonian appears in the discussion of the quantum transport through nanoscale conductors [30]. Hamiltonian (1) also appears in models describing the dynamics of cold atoms in optical lattices in tight-binding approximation [31].

In the present work we elucidate the similarity between the Hamiltonian (1) with infinite boundary conditions, i.e.,  $-\infty \leq j \leq \infty$ , and the Hamiltonian describing the discrete-charge mesoscopic quantum circuits [32,33,34]. In particular, we show that the Hamiltonian (1) with Dirichlet boundary conditions can be considered as a realization of the deformed parafermionic polynomial algebras. Parafermions of order  $p$  (with  $p$  being a positive integer) have been introduced in [35,36,37]. The nature of these particles is such that at most  $p$  identical particles can be found in the same state. The usual spin half fermions correspond to  $p = 1$ . The notion of parafermionic algebra has been extended by Quesne [38]. The relation between parafermionic algebras and other algebras has been given in [39,40,41].

Some extensions of the Hamiltonian (1) are in order. For example one can add an aperiodic potential to the Hamiltonian, see for example [42] or add non linear terms as in pendulum model [43].

The layout of the paper is as follows: In section 2, we introduce the model and discuss in summary the discrete-charge mesoscopic quantum circuits. In section 3, periodic boundary condition is discussed. In section 4, a realization of the deformed parafermionic polynomial algebra is presented. In section 5, the

time-evolution operator for a finite system with Dirichlet and periodic boundary conditions is discussed. In section 6, we present a summary of our results.

## 2 The model

The system under study is the derived quantum motion of a charged particle in a one-dimensional array of single-band quantum wells in tight-binding approximation and under the action of an arbitrary time-dependent external field  $F(t)$ .

The Hamiltonian of such a system can be written as

$$\hat{H} = G \sum_j (|j\rangle\langle j+1| + |j+1\rangle\langle j|) + F(t) \sum_j j|j\rangle\langle j|, \quad (2)$$

where the ket  $|j\rangle$  represents a Wannier state located on site  $j$ . These states fulfill the orthogonality condition  $\langle j|j'\rangle = \delta_{jj'}$ . The force  $F(t)$  is an arbitrary time-dependent force and the real parameter  $G$  is the nearest-neighbors coupling strength. In the following we will investigate the Hamiltonian (2) and its applications in derived discrete-charge quantum circuits and specially we will show that this Hamiltonian, with closed boundary conditions, can be considered as a realization of the generalized parafermionic polynomial algebras.

### 2.1 Infinite chain

As the first boundary condition let us assume that the Hamiltonian (2) describes an infinite chain of single-state quantum wells, i.e.,  $-\infty < j < +\infty$ . In this case the Hamiltonian (2) can be written as [27]

$$\hat{H} = G(K + K^\dagger) + F(t)\hat{N}, \quad (3)$$

where

$$\begin{aligned} \hat{K} &= \sum_{j=-\infty}^{\infty} |j\rangle\langle j+1|, \\ \hat{K}^\dagger &= \sum_{j=-\infty}^{\infty} |j+1\rangle\langle j|, \\ \hat{N} &= \sum_{j=-\infty}^{\infty} j|j\rangle\langle j|. \end{aligned} \quad (4)$$

Operators  $K$  and  $K^\dagger$  act as ladder operators on the Wannier states

$$\begin{aligned} \hat{K}|j\rangle &= |j-1\rangle, \\ \hat{K}^\dagger|j\rangle &= |j+1\rangle, \end{aligned} \quad (5)$$

and the operator  $\hat{N}$  acts on the Wannier states as the position operator

$$\hat{N}|j\rangle = j|j\rangle. \quad (6)$$

These operators fulfill the following algebra

$$\begin{aligned} [\hat{N}, \hat{K}] &= -\hat{K}, \\ [\hat{N}, \hat{K}^\dagger] &= \hat{K}^\dagger, \\ [\hat{K}^\dagger, \hat{K}] &= 0. \end{aligned} \quad (7)$$

A quantum theory for mesoscopic circuits was proposed by Li and Chen [32] where charge discreteness was considered explicitly. In this case the charge operator  $\hat{q}$  takes on discrete eigenvalues when acting on the charge eigenvectors

$$\hat{q}|n\rangle = nq_e|n\rangle, \quad (8)$$

where  $n \in \mathcal{Z}$  (set of integers) and  $q_e$  is the electron charge. The charge operator  $\hat{q}$  can be represented in terms of the localized states as

$$\hat{q} = q_e \sum_{j=-\infty}^{\infty} j|j\rangle\langle j| = q_e \hat{N}. \quad (9)$$

Discrete-charge mesoscopic quantum circuits are described by the Hamiltonian

$$\begin{aligned} \hat{H} &= -\frac{\hbar^2}{2Lq_e^2}(\hat{Q} + \hat{Q}^\dagger - 2) + V(\hat{q}), \\ \hat{Q} &= e\frac{i}{\hbar}q_e\hat{p}, \\ [\hat{q}, \hat{p}] &= i\hbar, \end{aligned} \quad (10)$$

where  $L$  is the inductance of the circuit. The operators  $\hat{Q}$  and  $\hat{Q}^\dagger$  are charge ladder operators which together with the charge operator  $\hat{q}$  fulfill the following commutation relations

$$\begin{aligned} [\hat{q}, \hat{Q}] &= -q_e\hat{Q}, \\ [\hat{q}, \hat{Q}^\dagger] &= q_e\hat{Q}^\dagger, \\ [\hat{Q}^\dagger, \hat{Q}] &= 0. \end{aligned} \quad (11)$$

Therefore the operators  $\hat{N} = q_e^{-1}\hat{q}$ ,  $\hat{Q}$  and  $\hat{Q}^\dagger$  satisfy the same algebra (7). When  $V(\hat{q}) = \frac{\hat{q}^2}{2C}$ , that is when there is a capacitor in the circuit with capacity  $C$ , we have a  $LC$ -design and for  $V(\hat{q}) = 0$ , we have a pure  $L$ -design.

The time-dependent Hamiltonian of a L-design circuit in charge representation and under the influence of an external potential  $\epsilon(t)$  is [32]

$$\hat{H} = -\frac{\hbar^2}{2Lq_e^2}(\hat{Q} + \hat{Q}^\dagger) + \epsilon(t)\hat{q}, \quad (12)$$

which is mathematically equivalent to the Hamiltonian (3).

### 3 Periodic boundary condition

We can also take the periodic boundary condition i.e,  $|n + N\rangle = |n\rangle$ . This kind of boundary condition appears for example in describing a circular quantum wire in tight-binding approximation. Where, a time-dependent magnetic field piercing the ring, acts as a driving force according to the Faraday's law. In this case the Hamiltonian (2) and the corresponding operators are defined as

$$\hat{H} = G \sum_{j=1}^N (|j\rangle\langle j+1| + |j+1\rangle\langle j|) + F(t) \sum_{j=1}^N j|j\rangle\langle j|, \quad (13)$$

$$\begin{aligned} \hat{K} &= \sum_{j=1}^N |j\rangle\langle j+1|, \\ \hat{K}^\dagger &= \sum_{j=1}^N |j+1\rangle\langle j|, \\ \hat{N} &= \sum_{j=1}^N j|j\rangle\langle j|. \end{aligned} \quad (14)$$

The commutation relations are now a deformation of the algebra (7) as

$$\begin{aligned} [\hat{N}, \hat{K}] &= -\hat{K}(1 - h(\hat{N})), \\ [\hat{N}, \hat{K}^\dagger] &= (1 - h(\hat{N}))\hat{K}^\dagger, \\ [\hat{K}, \hat{K}^\dagger] &= 0, \end{aligned} \quad (15)$$

where the polynomial operator  $h(\hat{N})$  is the deformation operator defined by

$$h(\hat{N}) = \frac{N(-1)^{N-1}}{(N-1)!} \prod_{j=2}^N (\hat{N} - j) \equiv N|1\rangle\langle 1|. \quad (16)$$

## 4 Deformed parafermionic algebra

In this section we show that the Hamiltonian (1), with Dirichlet boundary conditions, can be considered as a realization of the deformed parafermionic polynomial algebra. The Hamiltonian (1) appears for example in the dynamics of cold atoms in optical lattices [31,44] and also in the modeling of a finite quantum wire, in tight-binding approximation and under an external time-dependent electric field [30]. The Hamiltonian can be written as

$$\begin{aligned}
\hat{H} &= G \sum_{j=1}^{N-1} (|j\rangle\langle j+1| + |j+1\rangle\langle j|) + F(t) \sum_{j=1}^N j|j\rangle\langle j|, \\
\hat{K} &= \sum_{j=1}^{N-1} |j\rangle\langle j+1|, \\
\hat{K}^\dagger &= \sum_{j=1}^{N-1} |j+1\rangle\langle j|, \\
\hat{N} &= \sum_{j=1}^N j|j\rangle\langle j|.
\end{aligned} \tag{17}$$

The Hamiltonian in terms of the generators can be rewritten as

$$\hat{H}(t) = G(\hat{K} + \hat{K}^\dagger) + F(t)\hat{N}, \tag{18}$$

the new commutation relations are now as follows

$$\begin{aligned}
[\hat{N}, \hat{K}] &= -\hat{K}, \\
[\hat{N}, \hat{K}^\dagger] &= \hat{K}^\dagger, \\
[\hat{K}^\dagger, \hat{K}] &= |N\rangle\langle N| - |1\rangle\langle 1|.
\end{aligned} \tag{19}$$

From (17), it is clear that the operators  $\hat{K}$  and  $\hat{K}^\dagger$  are nilpotent of order  $N$ , that is,

$$\hat{K}^N = \hat{K}^{\dagger N} = 0. \tag{20}$$

The last commutation relation in (19) can be expressed as a polynomial in terms of the operator  $\hat{N}$ . This polynomial is not necessarily unique and we can choose a polynomial with the smallest degree. For this purpose let the unknown polynomial be  $f(\hat{N})$  then

$$f(\hat{N}) = |N\rangle\langle N| - |1\rangle\langle 1|. \tag{21}$$

The polynomial  $f(\hat{N})$  can be determined from (21) using the following conditions

$$\begin{aligned} f(\hat{N})|n\rangle &= 0, & \text{for } 2 \leq n \leq N-1, \\ f(\hat{N})|N\rangle &= |N\rangle, \\ f(\hat{N})|1\rangle &= -|1\rangle, \end{aligned} \tag{22}$$

after some simple algebra we find

$$\begin{aligned} f(\hat{N}) &= \frac{1}{(N-2)!} \prod_{j=2}^{N-1} (\hat{N} - j), & \text{for odd } N \geq 3, \\ f(\hat{N}) &= \frac{1}{(N-1)!} (2\hat{N} - (N+1)) \prod_{j=2}^{N-1} (\hat{N} - j), & \text{for even } N > 2, \\ f(\hat{N}) &= (2\hat{N} - 3) & \text{for } N = 2. \end{aligned} \tag{23}$$

Let us define the polynomial operator  $g(\hat{N})$  as

$$f(\hat{N}) = g(\hat{N} + 1) - g(\hat{N}). \tag{24}$$

From (23) one can find  $g(\hat{N})$  up to a constant as follows

$$\begin{aligned} g(\hat{N}) &= \frac{1}{(N-1)!} \prod_{j=2}^N (\hat{N} - j), & \text{for odd } N \geq 3, \\ g(\hat{N}) &= \left( \frac{2}{N!} \hat{N} - \frac{N+2}{N!} \right) \prod_{j=2}^N (\hat{N} - j), & \text{for even } N \geq 2. \end{aligned} \tag{25}$$

Putting everything together, we arrive at the following polynomial algebra

$$\begin{aligned}
[\hat{N}, \hat{K}^\dagger] &= \hat{K}^\dagger, \\
[\hat{N}, \hat{K}] &= -\hat{K}, \\
[\hat{K}^\dagger, \hat{K}] &= f(\hat{N}) = g(\hat{N} + 1) - g(\hat{N}), \\
[\hat{K}^\dagger \hat{K}, \hat{K} \hat{K}^\dagger] &= 0, \\
\hat{K}^N &= \hat{K}^{\dagger N} = 0, \\
\hat{K}^\dagger \hat{K} &= 1 - g(\hat{N}) =: \phi(\hat{N}), \\
\hat{K} \hat{K}^\dagger &= 1 - g(\hat{N} + 1) =: \phi(\hat{N} + 1),
\end{aligned} \tag{26}$$

where

$$\phi(1) = \phi(N + 1) = 0, \tag{27}$$

as is clear from the definition of the operators  $\hat{K}$  and  $\hat{K}^\dagger$ . Now we show that the polynomial algebra (26) is in fact a deformation of the usual parafermionic polynomial algebra  $\mathcal{L} = \{\hat{B}, \hat{B}^\dagger, \hat{M}\}$  [35] defined by

$$\begin{aligned}
[\hat{M}, \hat{B}] &= -\hat{B}, \\
[\hat{M}, \hat{B}^\dagger] &= \hat{B}^\dagger, \\
\hat{B}^{p+1} &= \hat{B}^{\dagger p+1} = 0, \\
\hat{B}^\dagger \hat{B} &= \hat{M}(p + 1 - \hat{M}) = [\hat{M}], \\
\hat{B} \hat{B}^\dagger &= (\hat{M} + 1)(p - \hat{M}) = [\hat{M} + 1], \\
\hat{M} &= \frac{1}{2}([\hat{B}^\dagger, \hat{B}] + p),
\end{aligned} \tag{28}$$

where  $p$  is a nonzero positive integer. For this purpose let us define the operators  $\hat{B}$ ,  $\hat{B}^\dagger$  and  $\hat{M}$  as

$$\begin{aligned}
\hat{B} &= \sqrt{\hat{N}(N - \hat{N})} \hat{K}, \\
\hat{B}^\dagger &= \hat{K}^\dagger \sqrt{\hat{N}(N - \hat{N})}, \\
\hat{M} &= \hat{N} - 1,
\end{aligned} \tag{29}$$

then it can be easily shown that the operators  $\hat{B}$ ,  $\hat{B}^\dagger$  and  $\hat{M}$ , fulfill the same algebra (28). It is interesting that for the special cases  $N = 2, 3$  the algebra (26) coincides with (28) and for  $N \geq 4$ , it is a deformation of the



algebra (28) with the deformation factor  $\sqrt{\hat{N}(N - \hat{N})}$  [38]. The deformation factor is not invertible since  $\sqrt{\hat{N}(N - \hat{N})}|N\rangle = 0$  but we can make it invertible by introducing an infinitesimal parameter  $\epsilon$  and defining a new operator  $\hat{D}_\epsilon := \sqrt{\hat{N}(N + \epsilon - \hat{N})} = \sqrt{(\hat{M} + 1)(N - 1 + \epsilon - \hat{M})}$  which tends to the deformation factor when  $\epsilon \rightarrow 0$ . Now the Hamiltonian can be written in terms of the generators of the parafermionic polynomial algebra (28) as

$$\hat{H} = G(\hat{D}_\epsilon^{-1}(\hat{M})\hat{B} + \hat{B}^\dagger\hat{D}_\epsilon^{-1}(\hat{M})) + F(t)(\hat{M} + 1). \quad (30)$$

The propagator for the non deformed Hamiltonian

$$\hat{H}^* = G(\hat{B} + \hat{B}^\dagger) + F(t)(\hat{M} + 1), \quad (31)$$

can be obtained exactly from the dynamical lie algebra method [45].

#### 4.1 The $su(2)$ representation of the special cases $N = 2, 3$

Setting  $N = 2$  in equations (26), (23) we arrive at

$$\begin{aligned} [\hat{N}, \hat{K}] &= -\hat{K}, \\ [\hat{N}, \hat{K}^\dagger] &= \hat{K}^\dagger, \\ [\hat{K}^\dagger, \hat{K}] &= 2\hat{N} - 3, \\ \hat{K}^2 &= \hat{K}^{\dagger 2} = 0, \\ \hat{N} &= 1 + \hat{K}^\dagger \hat{K}. \end{aligned} \quad (32)$$

From  $\{\hat{K}, \hat{K}^\dagger\} = I$  and  $\hat{K}^2 = \hat{K}^{\dagger 2} = 0$ , it is clear that the set  $\{\hat{K}, \hat{K}^\dagger, I\}$  is a realization of the usual spin-half fermionic algebra. By defining

$$\begin{aligned} \hat{J}_0 &= \hat{N} + \frac{3}{2}, \\ \hat{J}_+ &= \hat{K}^\dagger, \\ \hat{J}_- &= \hat{K}, \end{aligned} \quad (33)$$

we find

$$[\hat{J}_0, \hat{J}_\pm] = \pm \hat{J}_\pm, \quad [\hat{J}_+, \hat{J}_-] = 2\hat{J}_0, \quad (34)$$

which is the usual  $su(2)$  algebra. In terms of  $\hat{J}_\pm$  and  $\hat{J}_0$ , the Hamiltonian (18) can be rewritten as

$$\hat{H}(t) = G(\hat{J}_+ + \hat{J}_-) + F(t)(\hat{J}_0 - \frac{3}{2}). \quad (35)$$

For this Hamiltonian the Casimir operator  $\hat{J}^2 = \hat{J}_0^2 + \frac{1}{2}(\hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+)$  is a constant of motion.

Setting  $N = 3$  in equations (26), (23) we find

$$\begin{aligned} [\hat{N}, \hat{K}] &= -\hat{K}, \\ [\hat{N}, \hat{K}^\dagger] &= \hat{K}^\dagger, \\ [\hat{K}^\dagger, \hat{K}] &= \hat{N} - 2, \\ \hat{K}^3 &= \hat{K}^{\dagger 3} = 0, \\ \hat{N} &= 1 + \hat{K}^\dagger \hat{K} + \hat{K}^{\dagger 2} \hat{K}^2, \end{aligned} \quad (36)$$

again we can define the  $su(2)$  generators as

$$\begin{aligned} \hat{J}_+ &= \sqrt{2} \hat{K}^\dagger, \\ \hat{J}_- &= \sqrt{2} \hat{K}, \\ \hat{J}_0 &= \hat{N} - 2. \end{aligned} \quad (37)$$

The Hamiltonian in terms of this generators becomes

$$\hat{H}(t) = \frac{G}{\sqrt{2}}(\hat{J}_+ + \hat{J}_-) + F(t)(\hat{J}_0 - 2). \quad (38)$$

For  $N = 4, 5$ ,  $f(\hat{N})$  is a cubic polynomial operator (like Higgs algebra) and in general, for  $N = n, n+1$ ,  $f(\hat{N})$  is a polynomial operator of degree  $n-1$ .

## 5 Time-evolution operator

For  $N = 2, 3$  Hamiltonian has  $su(2)$  symmetry and the time-evolution operator can be obtained exactly. In this case the time-evolution operator  $U(t)$  can be written as

$$U(t) = e^{-\frac{i}{\hbar} \mu(t)} e^{-\frac{i}{\hbar} f(t) \hat{J}_0} e^{-\frac{i}{\hbar} g(t) \hat{J}_+} e^{-\frac{i}{\hbar} k(t) \hat{J}_-}. \quad (39)$$

Inserting the above expression in the following evolution equation

$$i\hbar \frac{d}{dt} U(t) = \hat{H} U(t), \quad (40)$$

we can find  $U(t)$  straightforwardly [45].

For  $N \geq 4$ , the previous method is not successful and we find a perturbative expansion for  $\hat{U}(t)$  in this case. For this purpose let us write the time-evolution operator as

$$U(t) = e^{-\frac{i}{\hbar}\varphi(t)\hat{N}}V(t), \quad (41)$$

where  $\varphi(t) = \int_0^t F(t')dt'$ . Substituting Eq.(41) into the evolution equation

$$i\hbar\frac{d}{dt}U(t) = \hat{H}U(t), \quad (42)$$

we find the following equation for  $V(t)$

$$i\hbar\frac{d}{dt}V(t) = G(e^{-\frac{i}{\hbar}\varphi(t)}\hat{K} + e^{\frac{i}{\hbar}\varphi(t)}\hat{K}^\dagger)V(t), \quad (43)$$

where the following relations have been used

$$\begin{aligned} e^{\frac{i}{\hbar}\varphi(t)\hat{N}}\hat{K}e^{-\frac{i}{\hbar}\varphi(t)\hat{N}} &= e^{-\frac{i}{\hbar}\varphi(t)}\hat{K}, \\ e^{\frac{i}{\hbar}\varphi(t)\hat{N}}\hat{K}^\dagger e^{-\frac{i}{\hbar}\varphi(t)\hat{N}} &= e^{\frac{i}{\hbar}\varphi(t)}\hat{K}^\dagger. \end{aligned} \quad (44)$$

The relations (44) can be obtained directly from the Baker-Hausdorf formula. The Eq.(43) can be written in terms of the state vector  $|\psi(t)\rangle = V(t)|\psi(0)\rangle$  as

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = G(e^{-\frac{i}{\hbar}\varphi(t)}\hat{K} + e^{\frac{i}{\hbar}\varphi(t)}\hat{K}^\dagger)|\psi(t)\rangle. \quad (45)$$

The new Hamiltonian

$$G(e^{-\frac{i}{\hbar}\varphi(t)}\hat{K} + e^{\frac{i}{\hbar}\varphi(t)}\hat{K}^\dagger), \quad (46)$$

has the following eigenvalues and eigenvectors respectively

$$\begin{aligned} \lambda_m &= 2G \cos \omega_m, & m &= 1, 2 \cdots N, \\ |\lambda_m, t\rangle &= d_m \sum_{n=1}^N e^{in\varphi(t)} \sin(n\omega_m) |n\rangle, \end{aligned} \quad (47)$$

where  $\omega_m = \frac{m\pi}{N+1}$  and  $d_m = \frac{1}{\sqrt{\sum_{n=1}^N \sin^2 n(\omega_m)}}$ , is a normalization coefficient.

Therefore, the eigenvalues are independent from the driving force  $F(t)$  but the eigenvectors are time dependent and make an orthonormal complete basis at any time. Now let us expand the state vector as

$$|\psi(t)\rangle = \sum_{\ell=1}^N C_\ell(t) |\lambda_\ell, t\rangle. \quad (48)$$

Inserting (48) into (45) we find

$$i\hbar\dot{C}_\ell(t) = \hbar\dot{\varphi} \sum_{\ell'=1}^N A_{\ell\ell'} C_{\ell'}(t) + \lambda_\ell(t) C_\ell(t). \quad (49)$$

where

$$A_{\ell\ell'} = \langle \lambda_\ell(t) | \frac{d}{dt} | \lambda_{\ell'}(t) \rangle = \frac{\sum_{n=1}^N n \sin(n\omega_\ell) \sin(n\omega_{\ell'})}{\sqrt{\sum_{n=1}^N \sin^2(n\omega_\ell) \sum_{n=1}^N \sin^2(n\omega_{\ell'})}} \quad (50)$$

are the elements of a symmetric matrix  $\hat{A}$ . The Eq.(49) can be written in matrix form as

$$i\hbar\dot{C}(t) = \hbar\dot{\varphi}\hat{A}C(t) + \hat{\lambda}C(t). \quad (51)$$

where

$$C = \begin{bmatrix} C_1(t) \\ C_2(t) \\ C_3(t) \\ \vdots \\ C_N(t) \end{bmatrix} \quad \hat{\lambda} = 2G \begin{bmatrix} \cos(\omega_1) & 0 & 0 & \cdots & 0 \\ 0 & \cos(\omega_2) & 0 & \cdots & 0 \\ 0 & 0 & \cos(\omega_3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cos(\omega_N) \end{bmatrix}. \quad (52)$$

By rescaling the parameters we can assume  $2G < 1$  and since  $\cos(\omega_m) < 1$ ,  $(1 \leq m \leq N)$ , so the matrix  $\hat{\lambda}$  has the property  $\lim_{k \rightarrow \infty} \hat{\lambda}^k \rightarrow 0$ . Now the equation (51) is similar to the Schrödinger equation with the perturbation  $\hat{\lambda}$ . Therefore the solution of (51) can be written as a series expansion in terms of the matrix  $\hat{\lambda}$  as

$$\hat{C}(t) = C^0(t) + \hat{\lambda}C^1(t) + \hat{\lambda}^2C^2(t) + \hat{\lambda}^3C^3(t) + \cdots, \quad (53)$$

substituting (53) into (51) and equating the terms of the same order in  $\hat{\lambda}$  we find

$$\begin{aligned} i\hbar\dot{C}^0(t) &= \hbar\dot{\varphi}\hat{A}C^0(0), \\ i\hbar\dot{C}^1(t) &= C^0(t) + \hbar\dot{\varphi}\hat{\lambda}^{-1}\hat{A}\hat{\lambda}C^1(t), \\ &\vdots \\ i\hbar\dot{C}^{(n)}(t) &= C^{(n-1)}(t) + \hbar\dot{\varphi}\hat{\lambda}^{-n}\hat{A}\hat{\lambda}^nC^{(n)}(t), \end{aligned} \quad (54)$$

The first equation leads to

$$C^{(0)}(t) = e^{-i\varphi(t)\hat{A}}C^{(0)}(0). \quad (55)$$

For  $n \geq 1$ , let us define  $\hat{\Delta}_n := \hat{\lambda}^{-n}\hat{A}\hat{\lambda}^n$ , then

$$\left(\frac{d}{dt} + i\dot{\varphi}\hat{\Delta}_n\right)C^n(t) = -\frac{i}{\hbar}C^{(n-1)}(t). \quad (56)$$

The equation(56) can be solved using the Green function method and treating  $C^{(n-1)}(t)$  as the source term for  $C^{(n)}(t)$  as

$$C^n(t) = \exp(-i\varphi(t)\hat{\Delta}_n)C^n(0) - \frac{i}{\hbar} \int_0^t \exp(-i\varphi(t-t')\hat{\Delta}_n)C^{(n-1)}(t')dt'. \quad (57)$$

As the boundary condition we can assume  $C(0) = C^0(0)$  and  $C^{(k)}(0) = 0$  for  $k \geq 1$ . So the equation (57) can be simplified as

$$C^n(t) = -\frac{i}{\hbar} \int_0^t \exp(-i\varphi(t-t')\hat{\Delta}_n)C^{(n-1)}(t')dt', \quad (n \geq 1). \quad (58)$$

Now having (55), we can find  $C^{(n)}(t)$  up to any order recursively.

## 6 Conclusions

The similarity between the Hamiltonian of a single-band tight-binding model and the discrete-charge mesoscopic quantum circuits is explicitly shown. It is shown that the former Hamiltonian, with Dirichlet boundary conditions, can be considered as a realization of the deformed parafermionic polynomial algebras.

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